5.6 Substitution in Definite Integrals - Area Between Curves

Example ① $\int_{-1}^{2} x^2 e^{x^3} dx = \int_{-1}^{8} \frac{1}{3} e^u du = \frac{1}{3} e^u \Big|_{-1}^{8} = \frac{1}{3}[e^8 - e^{-1}]$

$u = x^3$
$du = 3x^2 dx$
$\frac{1}{3} du = x^2 dx$

Change the limits!

Example ② $\int_{\pi/6}^{\pi/3} \cos x \, e^{\sin x} dx = \int_{1/2}^{\sqrt{3}/2} e^u du = e^u \Big|_{1/2}^{\sqrt{3}/2} = e^{\sqrt{3}/2} - e^{1/2}$

$u = \sin x$
$du = \cos x \, dx$

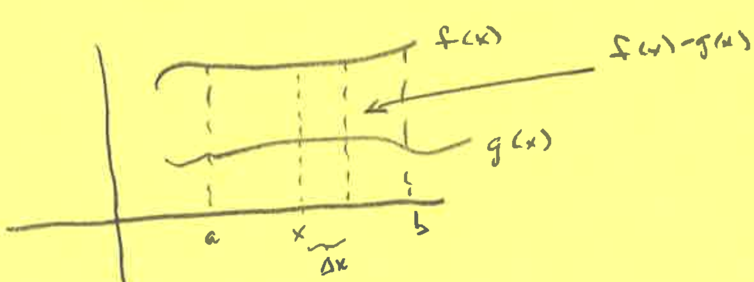
Even Functions $\int_{-a}^{a} f(x) dx = 2\int_{0}^{a} f(x) dx$

Odd Functions $\int_{-a}^{a} f(x) dx = 0$

Example ③ $\int_{-7}^{7} x^2 dx = 2\int_{0}^{7} x^2 dx = 2 \cdot \frac{x^3}{3} \Big|_0^7 = \frac{686}{3}$

Example ④ $\int_{-7}^{7} (x^5 + x^3) dx = 0$

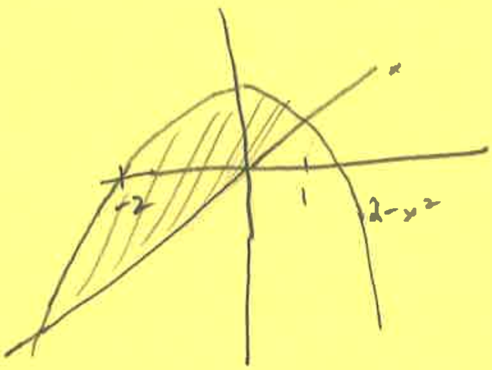
Area



Area of Rect $= [f(x) - g(x)] \Delta x$

Total Area $= \int_a^b [f(x) - g(x)] dx$

Example (5) Find the area between $f(x) = 2 - x^2$ and $g(x) = x$, $0 \le x \le 1$.



Find points of intersection:

$2 - x^2 = x$

$0 = x^2 + x - 2 = (x+2)(x-1)$

$x = -2$

$x = 1$

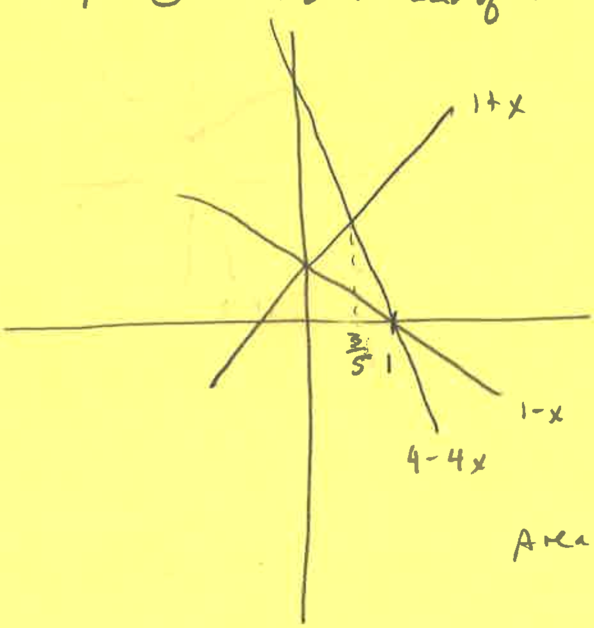
$$\text{Area} = \int_{-2}^{1} (2 - x^2 - x)\,dx$$

$$= 2x - \frac{x^3}{3} - \frac{x^2}{2} \Big|_{-2}^{1}$$

$$= 2 - 1/3 - 1/2 - \left[-4 + \frac{8}{3} - 2\right]$$

$$= 8 - 3 - 1/2 = 9/2$$

Example (6) Find the area of the region bounded by $y = 1 - x$, $y = 4 - 4x$, and $y = 1 + x$.



Find intersection points:

$1 - x = 1 + x$

$0 = 2x$

$x = 0$

$4 - 4x = 1 + x$

$3 = 5x$

$x = 3/5$

$1 - x = 4 - 4x$

$-3 = 3x$

$x = 1$

$$\text{Area} = \int_0^{3/5} [1 + x - (1 - x)]\,dx + \int_{3/5}^{1} [4 - 4x - (1 - x)]\,dx$$

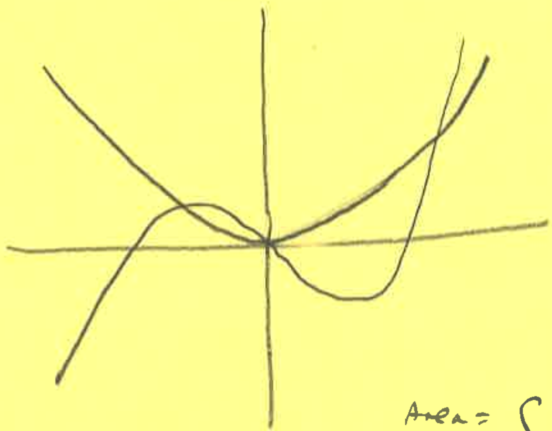
$$= \int_0^{3/5} 2x\,dx + \int_{3/5}^{1} (3 - 3x)\,dx$$

$$= \frac{2x^2}{2}\Big|_0^{3/5} + 3x - \frac{3}{2}x^2 \Big|_{3/5}^{1}$$

$$= \frac{9}{25} + \left(3 - \frac{3}{2}\right) - \left(\frac{9}{5} - \frac{27}{50}\right)$$

$$= \frac{3}{5}$$

Example 7 Find the area of the region bounded by $y = x^3 - 2x$ and $y = x^2$.



Find points of intersection:

$$x^3 - 2x = x^2$$

$$x^3 - 2x - x^2 = 0$$

$$x(x^2 - x - 2) = 0$$

$$x(x-2)(x+1) = 0$$

$$x = 0, -1, 2$$

$$\text{Area} = \int_{-1}^{0} (x^3 - 2x - x^2)\,dx + \int_{0}^{2} [x^2 - (x^3 - 2x)]\,dx$$

$$= \frac{x^4}{4} - x^2 - \frac{x^3}{3} \Big|_{-1}^{0} + \left(\frac{x^3}{3} - \frac{x^4}{4} + x^2\right)\Big|_{0}^{2}$$

$$= -\left(\frac{1}{4} - 1 + \frac{1}{3}\right) + \left(\frac{8}{3} - 4 + 4\right)$$

$$= \frac{37}{12}$$