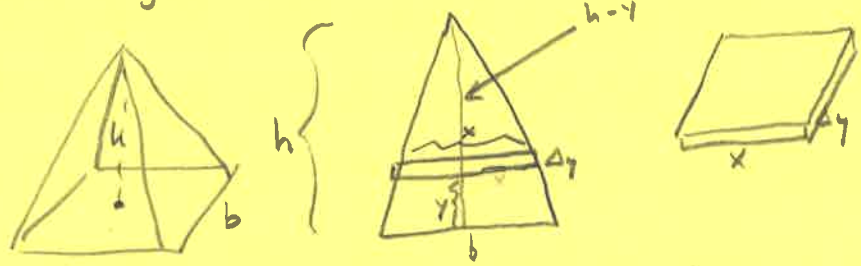
6.1 Volume with Cross Sections

Example 1 Find the volume of a pyramid with square base b of side b and height h.



$$\frac{h-y}{x/2} = \frac{h}{b/2}$$

$$b(h-y) = hx$$

$$x = \frac{b(h-y)}{h}$$

$$V \text{ of slice} = x^2 \Delta y = \left[\frac{b(h-y)}{h}\right]^2 \Delta y$$

$$\text{Volume} = \int_0^h \left[\frac{b(h-y)}{h}\right]^2 dy = \frac{b^2}{h^2}\int_0^h (h^2 - 2hy + y^2)\,dy$$

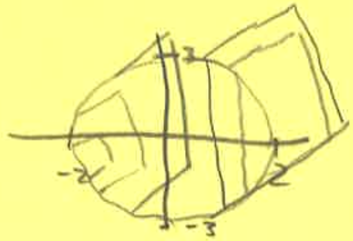
$$= \frac{b^2}{h^2}\left[h^2 y - hy^2 + \frac{y^3}{3}\right]\Big|_0^h$$

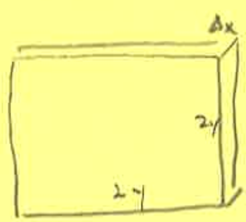
$$= \frac{b^2}{h^2}\left[h^3 - h^3 + \frac{h^3}{3}\right] = \frac{b^2 h}{3}$$

Cavalieri's Principle

Solids with equal heights and equal cross sections at each height have the same volume.

Example 2 A solid lies above the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Its cross sections perpendicular to the x-axis are squares. Find the volume.





Volume of slice $=$

$$4y^2 \Delta x = 4\left(9 - \frac{9x^2}{4}\right)\Delta x = (36 - 9x^2)\Delta x$$

$$\frac{y^2}{9} = 1 - \frac{x^2}{4}$$

$$y^2 = 9 - \frac{9x^2}{4}$$

Volume of slice $= \int_{-2}^{2} (36 - 9x^2)\, dx$

$$= 2\int_0^2 (36 - 9x^2)\, dx$$

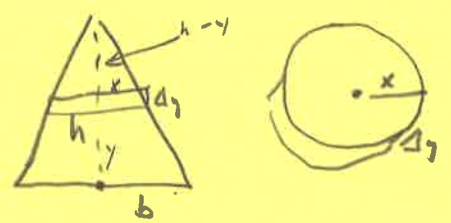
$$= 2\left[36x - 3x^3\right]\Big|_0^2$$

$$= 2\left[72 - 24\right]$$

$$= 96$$

Example ③ Find the volume of a cone with base radius b and height h.

method ① slicing



Volume of slice $= \pi x^2 \Delta y$

$\dfrac{h-y}{x} = \dfrac{h}{b}$ $\qquad = \pi\left(\frac{b}{h}(h-y)\right)^2 \Delta y$

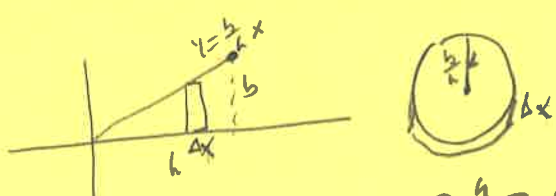
$x = \dfrac{b(h-y)}{h}$

$$\text{Volume} = \int_0^h \pi \frac{b^2}{h^2}(h-y)^2\, dy = \frac{\pi b^2}{h^2}\int_0^h (h^2 - 2hy + y^2)\, dy$$

$$= \frac{\pi b^2}{h^2}\left[h^2 y - hy^2 + \frac{y^3}{3}\right]\Big|_0^h$$

$$= \frac{\pi b^2}{h^2}\cdot\frac{h^3}{3} = \frac{\pi b^2 h}{3}$$

method ② Solid of revolution

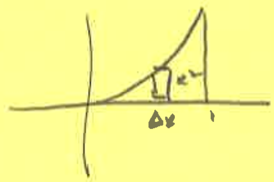


Volume of disk $= \pi\left(\frac{b}{h}x\right)^2 \Delta x$

$$\text{Volume} = \int_0^h \pi\left(\frac{b}{h}x\right)^2 dx = \frac{\pi b^2}{h^2}\int_0^h x^2\, dx$$

$$= \frac{\pi b^2}{h^2}\cdot\frac{x^3}{3}\Big|_0^h = \frac{\pi b^2 h}{3}$$

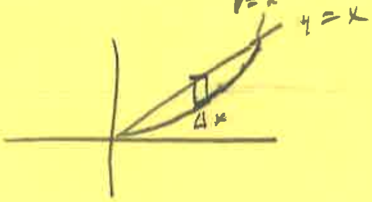
Example 4 The graph of $y = x^2, 0 \le x \le 1$ is rotated around the x-axis. Find the volume.



volume of disk $= \pi (x^2)^2 \Delta x$

$$\text{Volume} = \int_0^1 \pi x^4 \, dx = \frac{\pi x^5}{5}\Big|_0^1 = \frac{\pi}{5}$$

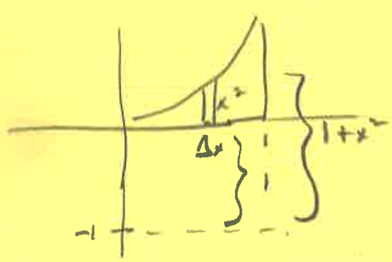
Example 5 The region between the graphs of $y = x^2$ and $y = x$, $0 \le x \le 1$ is rotated around the x-axis. Find the volume.



volume of washer $= \pi x^2 \Delta x - \pi x^4 \Delta x$
$= \pi (x^2 - x^4) \Delta x$

$$\text{Volume} = \int_0^1 \pi (x^2 - x^4) \, dx$$
$$= \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right)\Big|_0^1 = \pi \left(\frac{1}{3} - \frac{1}{5} \right)$$
$$= \frac{2}{15}\pi$$

Example 6 The graph of $y = x^2, 0 \le x \le 1$ is rotated around the line $x = -1$. Find the volume.



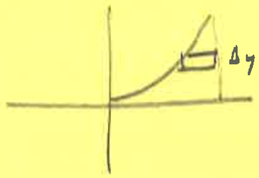
Volume of washer $=$
$\pi \left((1+x^2)^2 - 1^2 \right) \Delta x$

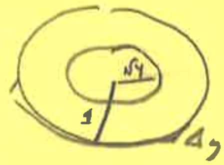
$$\text{Volume} = \int_0^1 \pi \left[(1+x^2)^2 - 1 \right] dx$$

$$= \pi \int_0^1 (1 + 2x^2 + x^4 - 1) \, dx$$
$$= \pi \left(\frac{2x^3}{3} + \frac{x^5}{5} \right)\Big|_0^1 = \pi \left(\frac{2}{3} + \frac{1}{5} \right)$$
$$= \frac{13}{15}\pi$$

Example 7 The graph of $y = x^2$, $0 \le x \le 1$ is rotated around the y-axis. Find the volume.

Solve for x in terms of y. $x = \sqrt{y}$





Volume of disk $= \pi\left(1^2 - (\sqrt{y})^2\right)\Delta y$

$$\text{Volume} = \int_0^1 \pi(1 - y)\,dy$$

$$= \pi\left(y - \frac{y^2}{2}\right)\Big|_0^1 = \pi/2$$