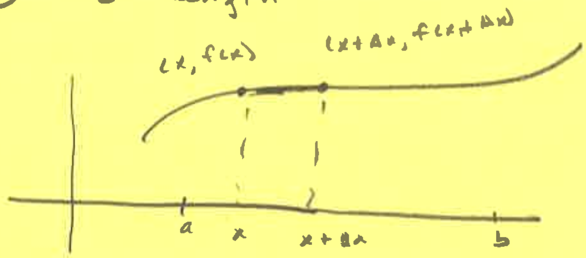
(6.3) Arc Length



length of segment $\approx \sqrt{(x+\Delta x - x)^2 + [f(x+\Delta x) - f(x)]^2}$

$= \sqrt{\Delta x^2 + [f(x+\Delta x) - f(x)]^2}$

$= \sqrt{1 + \left[\dfrac{f(x+\Delta x) - f(x)}{\Delta x}\right]^2}\ \Delta x$

$\approx \sqrt{1 + [f'(x)]^2}\ \Delta x$

$$\text{Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Example ① $f(x) = \dfrac{e^x + e^{-x}}{2}$ Find the length for $-1 \le x \le 1$

$f'(x) = \dfrac{e^x - e^{-x}}{2}$

$1 + [f'(x)]^2 = 1 + \dfrac{e^{2x} - 2 + e^{-2x}}{4} = \dfrac{4 + e^{2x} - 2 + e^{-2x}}{4}$

$= \dfrac{e^{2x} + 2 + e^{-2x}}{4} = \dfrac{(e^x + e^{-x})^2}{4}$

$\text{Length} = \int_{-1}^{1} \sqrt{\dfrac{(e^x + e^{-x})^2}{4}}\, dx = \int_{-1}^{1} \dfrac{e^x + e^{-x}}{2}\, dx$

$= \dfrac{e^x - e^{-x}}{2}\Big|_{-1}^{1} = \dfrac{e - \frac{1}{e}}{2} - \left[\dfrac{\frac{1}{e} - e}{2}\right]$

$= e - \frac{1}{e}$

Example 2 $f(x) = \frac{2}{3}x^{3/2} \quad 0 \le x \le 3$

$$f'(x) = x^{1/2}$$

$$\int_0^3 \sqrt{1+x}\,dx = \frac{2}{3}(1+x)^{3/2}\Big|_0^3 = \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right)$$

$$= \frac{14}{3}$$

Arc Length Function $\int_a^x \sqrt{1+[f'(t)]^2}\,dt$

Find the arc length function for $x^{2/3} + y^{2/3} = 1 \quad a \le t \le x$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{y^{2/3}}{x^{2/3}} = \frac{1-x^{2/3}}{x^{2/3}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1-x^{2/3}}{x^{2/3}}$$

$$= \frac{x^{2/3}+1-x^{2/3}}{x^{2/3}}$$

$$= \frac{1}{x^{2/3}}$$

$$\text{Length} = \int_a^x \sqrt{\frac{1}{t^{2/3}}}\,dt = \int_a^x t^{-1/3}\,dt = \frac{3}{2}t^{2/3}\Big|_a^x$$

$$= \frac{3}{2}\left(x^{2/3} - 0^{2/3}\right)$$