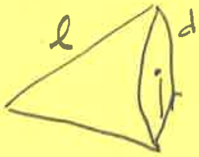
(6.4) Area of a Surface of Revolution

First: Area of a cone



l = slant height

r = radius

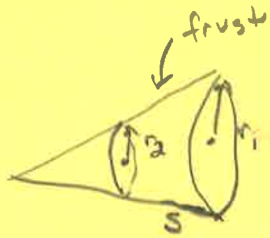
$d = 2\pi r$

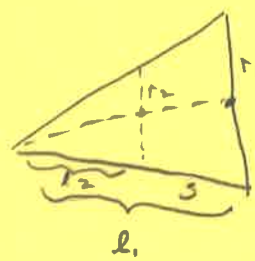
cut the cone down the side and flatten it:

$d = 2\pi r$

$$\theta = \frac{d}{2\pi l}(2\pi) = \frac{2\pi r}{l}$$

$$\text{Area} = \frac{\pi l^2 \cdot \theta}{2\pi} = \frac{l^2}{2} \cdot \frac{2\pi r}{l} = \pi r l$$





$s = l_1 - l_2$

$$\frac{l_1}{l_2} = \frac{r_1}{r_2} \qquad l_2 = \frac{l_1 r_2}{r_1}$$

$$s = l_1 - \frac{l_1 r_2}{r_1} = l_1\left(\frac{r_1 - r_2}{r_1}\right) \qquad l_2 = l_1 - s = \frac{s r_1}{r_1 - r_2} - s$$

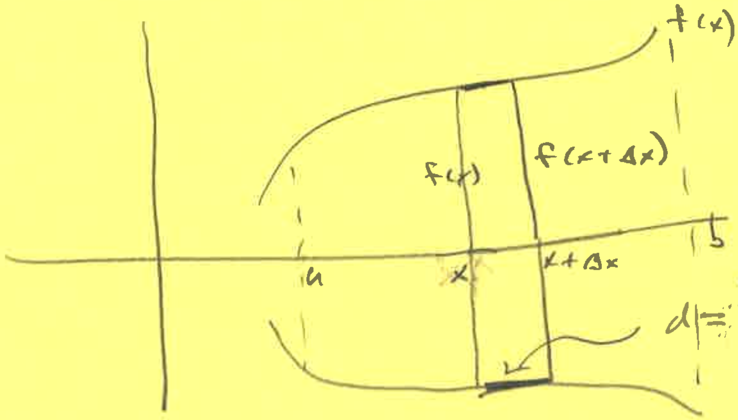
$$l_1 = \frac{s r_1}{r_1 - r_2} \qquad = s\left(\frac{r_1 - (r_1 - r_2)}{r_1 - r_2}\right)$$

$$= \frac{s r_2}{r_1 - r_2}$$

$$\text{Area} = \pi r_1 l_1 - \pi r_2 l_2$$

$$= \pi\left(\frac{s r_1^2}{r_1 - r_2} - s\frac{r_2^2}{r_1 - r_2}\right) = \pi s\left(\frac{r_1^2 - r_2^2}{r_1 - r_2}\right)$$

$$= \pi s (r_1 + r_2)$$



$$d = \sqrt{[f(x+\Delta x) - f(x)]^2 + \Delta x^2}$$

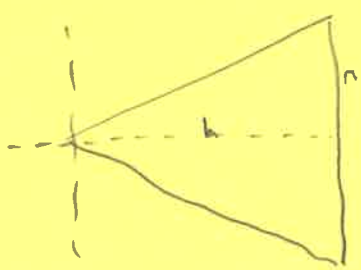
$$\text{Area of slice} \approx \pi \sqrt{[f(x+\Delta x) - f(x)]^2 + \Delta x^2}\,[f(x+\Delta x) + f(x)]$$

$$= \pi [f(x+\Delta x) + f(x)] \sqrt{\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]^2 + 1}\ \Delta x$$

$$\approx 2\pi f(x) \sqrt{1 + [f'(x)]^2}\ \Delta x$$

$$\text{Area} = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\ dx$$

Example ① Area of a cone



$$f(x) = \frac{r}{h} x$$

$$f'(x) = \frac{r}{h}$$

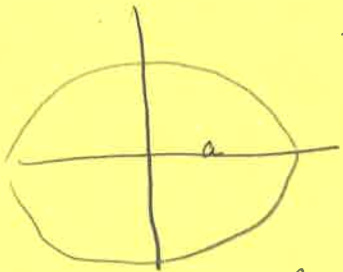
$$\text{Area} = \int_0^h 2\pi \frac{r}{h} x \sqrt{1 + \left(\frac{r}{h}\right)^2}\ dx$$

$$= \frac{\pi r}{h} x^2 \sqrt{1 + \left(\frac{r}{h}\right)^2}\ \Big|_0^h$$

$$= \pi r h \sqrt{1 + \left(\frac{r}{h}\right)^2}$$

$$= \pi r \sqrt{r^2 + h^2}$$

Example ② Area of a sphere



$$f(x) = \sqrt{a^2 - x^2}$$

$$f'(x) = \frac{1}{2\sqrt{a^2-x^2}} \cdot 2x = \frac{-x}{\sqrt{a^2-x^2}}$$

$$\text{Area} = \int_{-a}^{a} 2\pi \sqrt{a^2-x^2} \sqrt{1 + \frac{x^2}{a^2-x^2}}\, dx$$

$$= 4\pi \int_0^a \sqrt{a^2-x^2} \sqrt{\frac{a^2}{a^2-x^2}}\, dx$$

$$= 4\pi \int_0^a a\, dx$$

$$= 4\pi a^2$$

Example ③ Area of surface generated by rotating $x = \sqrt{y}$ around the y-axis, $0 \le y \le 2$.

$$f(y) = \sqrt{y}$$

$$f'(y) = \frac{1}{2\sqrt{y}}$$

$$\int_0^2 2\pi \sqrt{y} \sqrt{1 + \frac{1}{4y}}\, dy = \int_0^2 2\pi \sqrt{y + \frac{1}{4}}\, dy$$

$$= 2\pi \left(\frac{2}{3}\right) \left(y + \tfrac{1}{4}\right)^{3/2} \Big|_0^2 = \frac{4\pi}{3} \left(\left(\frac{9}{4}\right)^{3/2} - \left(\tfrac{1}{4}\right)^{3/2} \right)$$

$$= \frac{4\pi}{3} \left(\frac{27}{8} - \frac{1}{8} \right) = \frac{13\pi}{3}$$