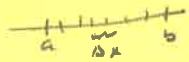
6.5 Work

Work = Force × Distance (constant force)

Type ① Object moving with force that varies with distance

Partition the path.  work for one segment is $F(x)\Delta x$. Total work is $\int_a^b F(x)\,dx$.

Example ① Calculate the work done in raising an object from sea level to an altitude of 10000m if the mass is 100kg

$$F = \frac{Gm_1m_2}{r^2}$$

The earth's radius is $\approx 6.3781 \times 10^6$ m

The earth's mass is $\approx 5.9724 \times 10^{24}$ kg

$$G \approx 6.6741 \times 10^{-11} \frac{m^3}{kg\,s^2}$$

$$W = \int_{r_1}^{r_2} G\frac{m_1m_2}{r^2}\,dr = -\frac{Gm_1m_2}{r}\Big|_{r_1}^{r_2}$$

$$= Gm_1m_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$\approx 6.6741 \times 10^{-11} \frac{m^3}{kg\,s^2}\left(5.9724 \times 10^{24}\,kg\right)(100)\,kg \left(\frac{1}{6.3781 \times 10^6\,m} - \frac{1}{6{,}388100\,m}\right)$$

$$\approx 9.6855 \times 10^6 \frac{kg\,m^2}{sec^2} \quad \text{(joules)}$$

Example 2 It takes 14 ft-lbs of work to stretch a spring from its equilibrium position a distance of 4 ft. Find the work required to stretch it from 4 ft to 5 ft.

Hooke's Law: the force is proportional to the distance stretched from equilibrium.

$F = kx$

$$14 = W = \int_0^4 kx\,dx = \frac{kx^2}{2}\Big|_0^4 = 8k$$

So $k = \frac{14}{8} = \frac{7}{4}$.

W from 4 to 5 is $\int_4^5 \frac{7}{4}x\,dx = \frac{7}{8}x^2\Big|_4^5$

$$= \frac{7}{8}(25-16) = \frac{7}{8}9 = \frac{63}{8} \text{ ft-lbs}$$

Example ③ A cylindrical tank of water is standing vertically. Its radius is 2 m and its height is 4 m. How much work does it take to pump the water to the top of the tank? The density of water is $\approx 1000 \text{ kg/m}^3$.

work = Force × Distance

joule = $m\,\text{kg}/\text{sec}^2$

$g = 9.8 \text{ m}/\text{sec}^2$

mass of slice = density × volume = $1000 \text{ kg}/\text{m}^3 \cdot \pi (4\text{m})^2 \Delta y$

$= 16000\pi \Delta y$

weight of slice = $g \cdot$ mass

$= 9.8 \cdot 16000\pi \Delta y$

$= 156800 \pi \Delta y$

work for slice = weight × distance

$= 156800\pi \Delta y \cdot 4-y$

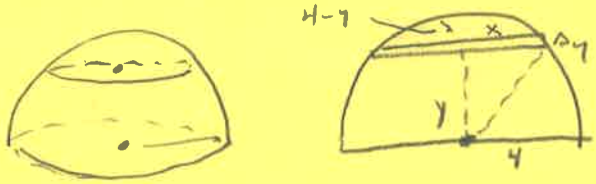
Total work $= 156800\pi \int_0^4 (4-y)\,dy$

$= 156800\pi \left(4y - \frac{y^2}{2}\right)\Big|_0^4$

$= 156800\pi\,(8)$

$= 1254400\pi$ joules

Example ④ A hemispherical tank with the flat surface down has a radius of 4 ft. It is filled with oil with a density of 50 lbs/ft³. Find the work required to pump the oil to a point 2 ft above the top of the tank.



$x^2+y^2=16$

$x=\sqrt{16-y^2}$

volume of slice $= \pi \cdot x^2 \Delta y = \pi(16-y^2)\Delta y$

weight of slice $= 50\pi(16-y^2)\Delta y$

work for slice $= (2+4-y)\cdot 50\pi(16-y^2)\Delta y$

Total work $= 50\pi\int_0^4 (6-y)(16-y^2)\,dy$

$= 50\pi\int_0^4 (96-16y-6y^2+y^3)\,dy$

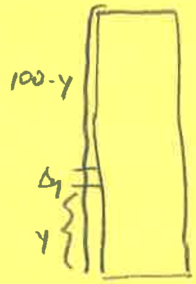
$= 50\pi\left(96y-8y^2-2y^3+\frac{y^4}{4}\right)\Big|_0^4$

$= 50\pi(384-128-128+64)$

$= 50\pi(192)$

$= 9600\pi$

Example 5 A chain weighs 1/2 lb per foot. It hangs over the side of a 100 ft tall building to the ground. How much work is required to pull it to the top?



weight of slice = $\frac{1}{2}\,\Delta y$

Distance of slice = $100-y$

Work for slice = $\frac{1}{2}(100-y)\,\Delta y$

$= (50 - y/2)\,\Delta y$

Total work = $\displaystyle\int_0^{100} (50 - y/2)\,dy$

$= 50y - \frac{y^2}{4}\Big|_0^{100}$

$= 5000 - \frac{10000}{4}$

$= 2500$ ft-lbs