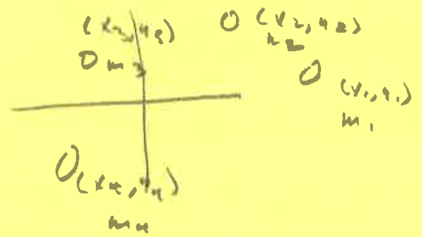
6.6 Centers of Mass



M_x = Moment about x-axis = mass·(signed distance to x-axis)

M_y = Moment about y-axis = mass·(signed distance to y-axis)

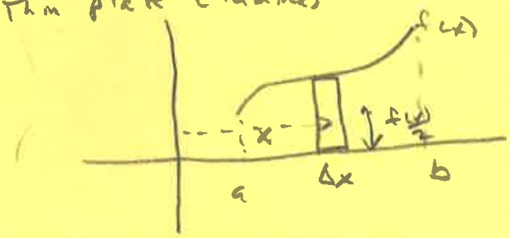
$(\bar{x}, \bar{y})$ = center of mass

M = total mass $= \sum_{i=1}^{n} m_i$

$M_x = \sum_{i=1}^{n} m_i y_i \qquad M_y = \sum_{i=1}^{n} m_i x_i$

δ = area density Mass = $\delta \cdot$ Area

Thin plate (lamina)



$M_y = \delta \cdot f(x)\,\Delta x \cdot x$

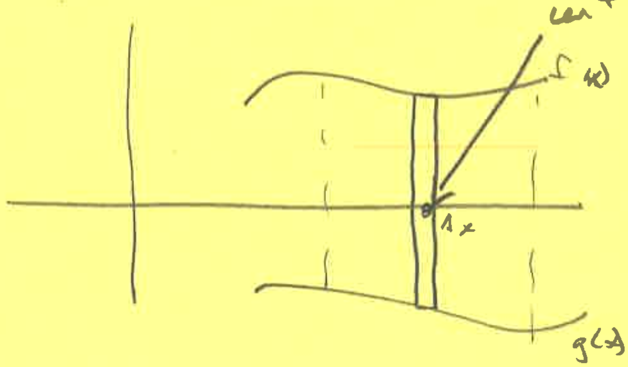
$M_x = \delta \cdot f(x)\,\Delta x \cdot \frac{f(x)}{2}$

Total $M_y = \int_a^b \delta(x)\, x f(x)\,dx$

Total $M_x = \int_a^b \delta(x) \frac{[f(x)]^2}{2}\,dx$

center $= \frac{f(x)+g(x)}{2}$

Rectangle



Area $= [f(x) - g(x)]\,\Delta x$

$M = \delta(x)[f(x) - g(x)]\,\Delta x$

$M_y = x\delta(x)[f(x) - g(x)]\,\Delta x$

$M_x = \delta(x)[f(x) - g(x)]\left[\frac{f(x)+g(x)}{2}\right]\Delta x$

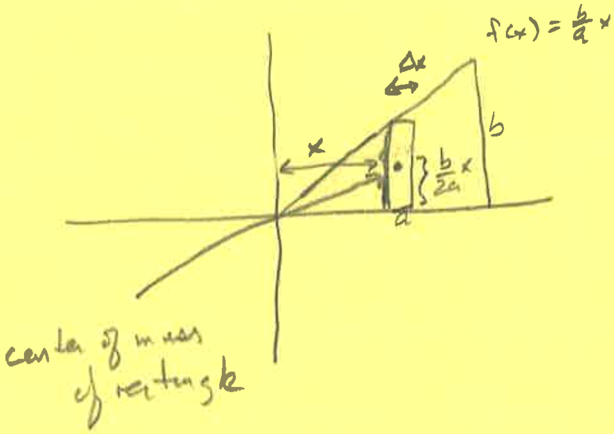
Total $M_y = \int_a^b x\delta(x)[f(x) - g(x)]\,dx \qquad \bar{x} = \frac{M_y}{M}$

$M_x = \int_a^b \delta(x)\left[\frac{f(x)^2 - g(x)^2}{2}\right]dx \qquad \bar{y} = \frac{M_x}{M}$

$$M = \int_a^b \delta(x)[f(x) - g(x)]\,dx$$

Centroid: $\delta = 1$.

Example ① Centroid of triangle



Rectangle: $M = \frac{b}{a}x\,\Delta x$

$$M_y = \frac{bx}{a}\Delta x \cdot x$$

$$M_x = \frac{bx}{a}\Delta x \left(\frac{b}{2a}x\right)$$

Triangle: $M = \int_0^a \frac{b}{a}x\,dy = \frac{b}{a}\left(\frac{x^2}{2}\right)\Big|_0^a$

$$= \frac{ab}{2}$$

$$M_y = \int_0^a b\frac{x^2}{a}\,dx = \frac{b}{a}\cdot\frac{x^3}{3}\Big|_0^a = \frac{ba^2}{3}$$

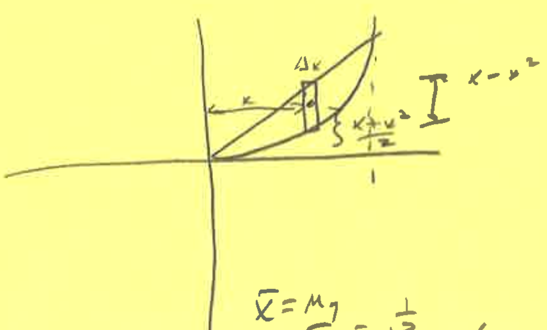
$$M_x = \int_0^a \frac{b^2x^2}{2a^2}\,dx = \frac{b^2}{2a^2}\frac{x^3}{3}\Big|_0^a$$

$$= \frac{b^2a}{6}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{ba^2}{3}}{\frac{ab}{2}} = \frac{2a}{3}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{b^2a}{6}}{\frac{ab}{2}} = \frac{b}{3}$$

Example ② Centroid of Region between graphs of $y = x^2$, $y = x$.



Rectangle: $M = (x - x^2)\Delta x$

$$M_y = (x - x^2)\Delta x \cdot x$$

$$M_x = (x - x^2)\Delta x\left(\frac{x + x^2}{2}\right)$$

Region: $M = \int_0^1 (x - x^2)\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = 1/6$

$$M_y = \int_0^1 (x^2 - x^3)\,dx = \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{12}$$

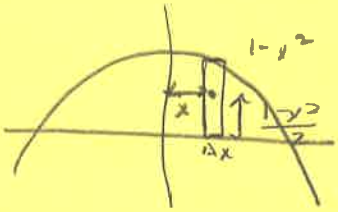
$$M_x = \int_0^1 \frac{x^2 - x^4}{2}\,dx = \frac{1}{2}\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1$$

$$= \frac{1}{2}\left(\frac{2}{15}\right) = \frac{1}{15}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{1}{12}}{\frac{1}{6}} = 1/2$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{1}{15}}{1/6} = \frac{6}{15} = \frac{2}{5}$$

Example ③ center of mass of region between $1-x^2$ and x-axis, $\delta(x)=x^2$ ③



$$M = x^2(1-x^2)\Delta x$$

$$M_x\,(\text{rect}) = x^2(1-x^2)\Delta x\left(\frac{1-x^2}{2}\right)$$

$$M_y\,(\text{rect}) = x^2(1-x^2)\Delta x \cdot x$$

$$M = \int_{-1}^{1} x^2(1-x^2)\,dx = 2\int_0^1 (x^2-x^4)\,dx = 2\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1 = \frac{4}{15}$$

$$M_y = \int_{-1}^{1} x^3(1-x^2)\,dx = 0 \quad \text{(odd function)}$$

$$M_x = \int_{-1}^{1} x^2(1-x^2)^2\,dx = 2\int_0^1 (x^2-2x^4+x^6)\,dx =$$

$$2\left(\frac{x^3}{3} - \frac{2x^5}{5} + \frac{x^7}{7}\right)\Big|_0^1 = 2\left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right)$$

$$= 2\left(\frac{35-21+15}{105}\right) = \frac{58}{105}$$

$$\bar{x} = \frac{M_y}{M} = 0 \qquad \bar{y} = \frac{\frac{58}{105}}{\frac{4}{15}} = \frac{29}{2(7)} = \frac{29}{14}$$