8.2 Trigonometric Integrals

(I) $\int \sin^m x \cos^n x \, dx$

(Ia) m is odd - split off 1 sin. Convert others to cos.

Example ① $\int \sin^3 x \cos^2 x \, dx = \int \sin x \sin^2 x \cos^2 x \, dx$

$$= \int \sin x \,(1-\cos^2 x)\cos^2 x \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\int (1-u^2)\, u^2 \, du$$

$$= -\int (u^2 - u^4)\, du$$

$$= -\frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

(Ib) m is even, n is odd - split off 1 cos. Convert others to sin.

Example ② $\int \sin^4 x \cos^3 x \, dx = \int \sin^4 x \,(1-\sin^2 x)\cos x \, dx$

$$= \int (\sin^4 x - \sin^6 x)\cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int (u^4 - u^6)\, du$$

$$= \frac{u^5}{5} - \frac{u^7}{7} + C$$

$$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$$

(I c) m, n both even Use $\sin^2 x = \frac{1-\cos 2x}{2}$ $\cos^2 x = \frac{1+\cos 2x}{2}$ ①

Example ③ $\int \sin^4 x \cos^2 x \, dx = \int \left[\frac{1-\cos 2x}{2}\right]^2 \left[\frac{1+\cos 2x}{2}\right] dx$

$= \frac{1}{8}\int (1 - 2\cos 2x + \cos^2 2x)(1 + \cos 2x)\, dx$

$= \frac{1}{8}\int (1 + \cos 2x - 2\cos 2x - 2\cos^2 2x + \cos^2 2x + \cos^3 2x)\, dx$

$= \frac{1}{8}\int (1 - \cos 2x - \cos^2 2x + \cos^3 2x)\, dx$

$= \frac{1}{8}\int \left[1 - \cos 2x - \left(\frac{1+\cos 4x}{2}\right) + \cos 2x\,(1 - \sin^2 2x)\right] dx$

$u = \sin 2x$

$du = 2\cos 2x$

$\frac{1}{2}\, du = \cos 2x$

$= \frac{1}{8}\left[\int \left(\frac{1}{2} - \frac{\cos 4x}{2}\right) dx - \int \frac{u^2}{2}\, du\right]$

$= \frac{1}{8}\left[\frac{x}{2} - \frac{\sin 4x}{8} - \frac{u^3}{6}\right] + C$

$= \frac{x}{16} - \frac{\sin 4x}{64} - \frac{\sin^3 2x}{48} + C$

(II) $\int \tan^m x \sec^n x \, dx$

(II a) n even, $n>0$ split off $\sec^2 x$, use $\sec^2 x = 1+\tan^2 x$

Example 4 $\int \tan^3 x \sec^4 x \, dx = \int \tan^3 x \,(1+\tan^2 x)\sec^2 x \, dx$

$$= \int (\tan^3 x + \tan^5 x)\sec^2 x \, dx$$

$u = \tan x$

$du = \sec^2 x \, dx$

$$= \int (u^3 + u^5)\, du$$

$$= \frac{u^4}{4} + \frac{u^6}{6} + C$$

$$= \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C$$

(II b) m odd, $n>0$ split off $\sec x \tan x$, use $\tan^2 x = \sec^2 x - 1$

Example 5 $\int \tan^3 x \sec^3 x \, dx = \int \sec x \tan x \,(\sec^2 x - 1)\sec^2 x \, dx$

$$= \int \sec x \tan x \,(\sec^4 x - \sec^2 x)\, dx$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$= \int (u^4 - u^2)\, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3}$$

(II c) n odd, m even use $\tan^2 x = \sec^2 x - 1$, use parts

Example 6 $\int \tan^2 x \sec x \, dx = \int (\sec^2 x - 1)\sec x \, dx$

$$= \int \sec^3 x \, dx - \int \sec x \, dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$u = \sec x \qquad du = \sec x \tan x$

$dv = \sec^2 x\,dx \qquad v = \tan x$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\,dx$$

$$= \sec x \tan x - \int \sec^3 x + \int \sec x\,dx$$

take to l.h.s.

$$2\int \sec^3 x = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\int \sec^3 x = \frac{\sec x \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x| + C$$

$$\int \tan^2 x \sec x\,dx = \frac{\sec x \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x| - \ln|\sec x + \tan x| + C$$

$$= \frac{\sec x \tan x}{2} - \frac{1}{2}\ln|\sec x + \tan x| + C$$

(III) Eliminating square roots. Use $\frac{1+\cos 2x}{2} = \cos^2 x$ or $\frac{1-\cos 2x}{2} = \sin^2 x$

Example 7
$$\int \sqrt{1+\cos 4x}\,dx = \int \sqrt{2}\sqrt{\frac{1+\cos 4x}{2}}\,dx$$

$$= \int \sqrt{2}\sqrt{\cos^2 2x}\,dx$$

$$= \sqrt{2}\int \cos 2x\,dx$$

$$= \sqrt{2}\,\frac{\sin 2x}{2} + C$$

$$= \frac{\sin 2x}{\sqrt{2}} + C$$

(IV) $\int \sin mx \sin nx \, dx \qquad \int \sin mx \cos nx \, dx \qquad \int \cos mx \cos nx \, dx$

use $\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$

$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$

$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$

Example 5 $\int \sin 3x \cos 5x \, dx = \int \frac{1}{2}\left[\sin(-2x) + \sin(8x)\right] dx$

$= \frac{1}{2}\int (\sin 8x - \sin 2x) \, dx$

$= \frac{1}{2}\left[\left(-\frac{\cos 8x}{8}\right) + \frac{\cos 2x}{2}\right] + C$

$= \frac{\cos 2x}{4} - \frac{\cos 8x}{16} + C$