(8.3) Trig Substitution

$\sqrt{a^2-x^2}$ use $x = a\sin\theta$

$\sqrt{a^2+x^2}$ use $x = a\tan\theta$

$\sqrt{x^2-a^2}$ use $x = a\sec\theta$

Example 1

$$\int \sqrt{a^2-x^2}\,dx = \int a^2\cos^2\theta\,d\theta = \int a^2\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$x = a\sin\theta$

$dx = a\cos\theta\,d\theta$

$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = a\cos\theta$

$$\frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) =$$

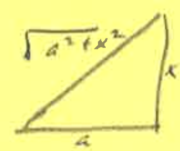
$$\frac{a^2\theta}{2} + \frac{a^2\sin 2\theta}{4}$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{a^2}{4}\cdot 2\sin\theta\cos\theta$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{a^2}{2}\cdot\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$$

Example 2



$$\int \sqrt{a^2+x^2}\,dx = \int a\sec\theta\cdot a\sec^2\theta\,d\theta$$

$$= \int a^2\sec^3\theta\,d\theta$$

$x = a\tan\theta$

$dx = a\sec^2\theta\,d\theta$

$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta}$

$= a\sqrt{1+\tan^2\theta}$

$= a\sec\theta$

$$= \frac{a^2}{2}\sec\theta\tan\theta - \frac{a^2}{4}\ln|\sec\theta+\tan\theta|$$

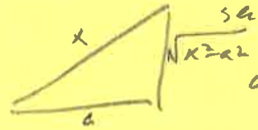
$$= \frac{a^2}{2}\cdot\frac{\sqrt{a^2+x^2}}{a}\cdot\frac{x}{a}$$

$$+ \frac{a^2}{2}\ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right|$$

$$= \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right|$$

$+ C$

Example ③ $\int \frac{1}{x^2\sqrt{x^2-a^2}}\,dx = \int \frac{1}{a^2\sec^2\theta \cdot a\tan\theta}\, a\sec\theta\tan\theta\, d\theta$



$\sec\theta = \frac{x}{a}$

$\cos\theta = \frac{a}{x}$

$x = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

$x^2 = a^2\sec^2\theta$

$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2}$

$= \sqrt{a^2(\sec^2\theta - 1)}$

$= a\sqrt{\tan^2\theta}$

$= a\tan\theta$

$= \frac{1}{a^2}\int \cos\theta\, d\theta$

$= \frac{1}{a^2}\sin\theta + C$

$= \frac{1}{a^2}\frac{\sqrt{x^2-a^2}}{x} + C$

Example ④ $\int \frac{\sqrt{1-x^2}}{x^4}\,dx = \int \frac{\cos\theta}{\sin^4\theta}\cdot\cos\theta\, d\theta$

$x = \sin\theta$

$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta}$

$= \cos\theta$

$dx = \cos\theta\, d\theta$

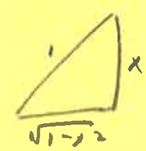
$= \int \frac{\cos^2\theta}{\sin^4\theta}\, d\theta$

$= \int \cot^2\theta\csc^2\theta\, d\theta$

$u = \cot\theta$

$du = -\csc^2\theta\, d\theta$

$= \int -u^2\, du$



$= -\frac{u^3}{3} + C$

$= -\cot^3\theta + C$

$= -\frac{(1-x^2)^{3/2}}{x^3} + C$

$= \frac{-(1-x^2)^{3/2}}{x^3} + C$