8.4 Partial Fractions

For integrating rational functions. $\frac{p(x)}{q(x)}$

Step 1 If degree $(p(x)) \geq$ degree of $(q(x))$, divide $q(x)$ into $p(x)$. The remainder term will have degree(denom) $\geq$ degree(numer).

Example 1 $\frac{x^2+1}{x^2-1}$

$$\begin{array}{r|ll} & 1 & \\ x^2-1 & x^2 & +1 \\ & x^2 & -1 \\ \hline & & 2 \end{array}$$

x^2-1 goes into x^2+1 1 time with a remainder of 2,

so $\frac{x^2+1}{x^2-1} = 1 + \frac{2}{x^2-1}$

Step 2 Factor the denominator. Split up the rational function into terms corresponding to each factor of the denominator.

Case 1 powers of linear terms $\frac{1}{(x-r)^n}$

Use $\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_n}{(x-r)^n}$

Case 2 powers of irreducible quadratic terms $\frac{1}{(ax^2+bx+c)^n}$

Use $\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$

Step 3 solve for the coefficients

Example 1 $\int \frac{1}{x^2-1}\,dx$: $\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$

Multiply both sides by $(x+1)(x-1)$

$1 = A(x-1) + B(x+1)$

$1 = (A+B)x + (-A+B)$

Equate coefficients: $A+B=0$

$$\begin{array}{r} -A+B=1 \\ \hline 2B=1 \\ 2A=-1 \end{array} \qquad \begin{array}{l} B=1/2 \\ A=-1/2 \end{array}$$

$\int \frac{1}{x^2-1}\,dx = \int \left(\frac{-1/2}{x+1} + \frac{1/2}{x-1} \right) dx = -1/2 \ln|x+1| + 1/2 \ln|x+1| + C$

Example ② $\displaystyle\int \frac{x+1}{x^3-2x^2+x}\,dx =$

$$\frac{x+1}{x^3-2x^2+x} = \frac{x+1}{x(x^2-2x+1)} = \frac{x+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)^2} + \frac{C}{x-1}$$

$$x+1 = A(x-1)^2 + Bx + Cx(x-1)$$

$$x+1 = (A+C)x^2 + (-2A+B-C)x + A$$

$A + C = 0$ $\qquad C = -1$

$-2A + B - C = 1$ $\qquad -2 + B + 1 = 1$

$A = 1$ $\qquad B = 2$

$$\int \frac{x+1}{x^3-2x^2+x}\,dx = \int\left[\frac{1}{x} + \frac{2}{(x-1)^2} + \frac{-1}{x-1}\right]dx$$

$$= \ln|x| - \frac{2}{x-1} - \ln|x-1| + C$$

Example ③ $\displaystyle\int \frac{x^4+x+1}{x^3-x}\,dx = \int x + \frac{x^2+x+1}{x^3-x}\,dx$

$$\begin{array}{r|l} & \quad x \\ x^3-x & x^4 + \quad x+1 \\ & x^4 \quad - x^2 \\ \hline & \quad x^2+x+1 \end{array}$$

$$= \frac{x^2}{2} + \int \frac{x^2+x+1}{x(x^2-1)}\,dx$$

$$= \frac{x^2}{2} + \int \frac{x^2+x+1}{x(x+1)(x-1)}\,dx$$

$$\frac{x^2+x+1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$$x^2+x+1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$$

subst. $x=0$: $\quad 1 = A(-1)$ $\quad A = -1$

subst. $x = 1$: $\quad 3 = C\cdot 2$ $\quad C = 3/2$

subst. $x=-1$: $\quad 1 = B\cdot 2$ $\quad B = 1/2$

$$= \frac{x^2}{2} + \int\left[\frac{-1}{x} + \frac{\frac{1}{2}}{x+1} + \frac{3/2}{x-1}\right]dx$$

$$= \frac{x^2}{2} - \ln|x| + \frac{1}{2}\ln|x+1| + 3/2\ln|x-1| + C$$

Example: $\displaystyle\int \frac{1}{x^2+4x+5}\,dx$

Roots: $\dfrac{-4 \pm \sqrt{16-4(1)(5)}}{2} = \dfrac{-4 \pm \sqrt{-4}}{2}$ so no real roots $\Rightarrow$ x^2+4x+5 is irreducible

so: complete the square

$$\int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{x^2+4x+4+1}\,dx$$

$$= \int \frac{1}{(x+2)^2+1}\,dx$$

$$= \tan^{-1}(x+2) + C$$

Example: $\displaystyle\int \frac{2x-5}{x^2-6x+13}\,dx = \int \frac{2x-5}{x-6x+9+4}\,dx$

$$= \int \frac{2x-5}{(x-3)^2+4}\,dx = \int \frac{2(x-3)+1}{(x-3)^2+4}\,dx$$

$u = (x-3)^2+4$

$du = 2(x-3)du$

$$= \int \frac{du}{u} + \int \frac{1}{(x-3)^2+4}\,dx$$

$$= \ln|u| + \frac{1}{2}\tan^{-1}\frac{x-3}{2} + C$$

$$= \ln\left[(x-3)^2+4\right] + \frac{1}{2}\tan^{-1}\frac{x-3}{2} + C$$

$$\int \frac{2/3\, x + 1/3}{x^2+x+1}\, dx = \int \frac{2/3\, x + 1/3}{x^2+x+\frac{1}{4} - 1/4}\, dx = \int \frac{2/3\, x + 1/3}{(x+1/2)^2 - 1/4}\, dx$$

$$= \frac{1}{3} \int \frac{2x+1}{(x+1/2)^2 - 1/4}\, dx = \frac{1}{3} \int \frac{2(x+1/2)}{(x+1/2)^2 - 1/4}\, dx = \frac{1}{3} \int \frac{du}{u} = \ln|u|$$

$u = (x+1/2)^2 - 1/4$

$du = 2(x+1/2)\, dx$

$$= \frac{1}{3} \ln (x^2+x+1)$$

$$\int \frac{1}{(x^2-x)(x^2+x+1)}\, dx = -\ln|x| + 1/3 \ln|x-1| + \frac{1}{3} \ln (x^2+x+1)$$

Partial Fraction Decomposition of $\frac{1}{(x-3)^4 + (x^2-2x+2)^3}$

$$\frac{A}{(x-3)^4} + \frac{B}{(x-3)^3} + \frac{C}{(x-3)^2} + \frac{D}{x-3} + \frac{Ex+F}{(x^2-2x+2)^3} + \frac{Gx+H}{(x^2-2x+2)^2} + \frac{Ix+J}{x^2-2x+2}$$

Example

$$\int \frac{1}{x^2(x+2)^2}\,dx$$

$$\frac{1}{x^2(x+2)^2} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{(x+2)^2} + \frac{D}{x+2}$$

$$1 = A(x+2)^2 + Bx(x+2)^2 + Cx^2 + D(x+2)x^2$$

$$1 = (B+D)x^3 + (A+4B+C+2D)x^2 + (2A+4B)x + 4A$$

$B + D = 0$

$A + 4B + C + 2D = 0$

$4A + 4B = 0$

$4A = 1$

$A = 1/4$

$4B = -4A$

$B = -A = \frac{-1}{4}$

$D = -B = \frac{1}{4}$

$C = -A - 4B - 2D = \frac{-1}{4} + 1 - \frac{1}{2} = \frac{1}{4}$

$$\int \frac{1}{x^2(x+2)^2}\,dx = \int \left[\frac{1/4}{x^2} - \frac{1/4}{x} + \frac{1/4}{(x+2)^2} + \frac{1/4}{x+2}\right] dx$$

$$= \frac{-1}{4x} - 1/4 \ln|x| - \frac{1}{4(x+2)} + \frac{1}{4}\ln|x+2| + C$$

Example $\displaystyle\int \frac{1}{(x^2-x)(x^2+x+1)}\,dx$

$$\frac{1}{(x^2-x)(x^2+x+1)} = \frac{1}{x(x-1)(x^2+x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1}$$

$$1 = A(x-1)(x^2+x+1) + Bx(x^2+x+1) + (Cx+D)x(x-1)$$

$x = 1$: $1 = B(1)(3)$ $B = 1/3$

$x = 0$: $1 = A(-1)(1)$ $A = -1$

$$1 = (A+B+C)x^3 + (B-C+D)x^2 + (B-D)x - A$$

$A + B + C = 0$

$B - C + D = 0$

$B - D = 0$

$A = -1$

$D = B = 1/3$

$C = B + D = 2/3$

$$\int \frac{1}{(x^2-x)(x^2+x+1)}\,dx = \int \frac{-1}{x} + \frac{1/3}{x-1} + \frac{2/3\,x + 1/3}{x^2+x+1}$$