8.6 Numerical Integration

Rectangles:

Divide into n segments

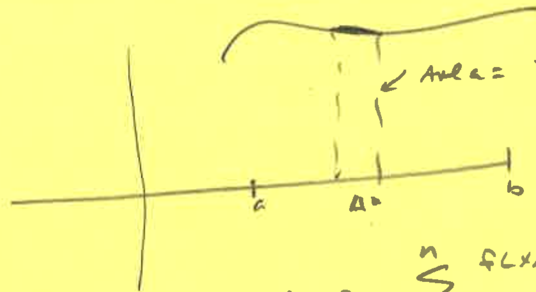
$\Delta x = \frac{b-a}{n}$

Area of rect $= f(x)\,\Delta x$

Total area $= \int_a^b f(x)\,dx$

Approximation to total area $= \sum_{i=1}^{n} f(x_i)\,\Delta x$

Trapezoidal Rule



Area $= \frac{f(x_i) + f(x_{i+1})}{2}\,\Delta x$

Approximation to total Area $= \sum_{i=1}^{n} \frac{f(x_i) + f(x_{i+1})}{2}\,\Delta x$

$$= \left[\frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2} + \dots + \frac{f(x_{n-2})+f(x_{n-1})}{2} + \frac{f(x_{n-1})+f(x_n)}{2}\right]\Delta x$$

$$= \frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2}$$

Example $f(x) = x^2 \qquad n = 4$

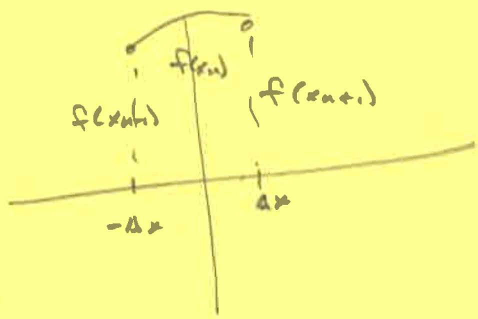
i	x_i	$f(x_i)$	coef	product
0	0	0	1	0
1	1/4	1/16	2	1/8
2	1/2	1/4	2	1/2
3	3/4	9/16	2	9/8
4	1	1	1	1

total $= 2 + \frac{10}{8} + \frac{4}{8} = \frac{30}{8} = \frac{15}{4}$

Approximation $= \frac{1}{2}\left(\frac{15}{4}\right) = \frac{15}{8}$

Simpson's Rule: requires n to be even.

translate region to the y-axis!



Approximation $g(x) = ax^2 + bx + c$

$$g(-\Delta x) = a\Delta x^2 - b\Delta x + c = f(x_{n-1})$$

$$g(\Delta x) = a\Delta x^2 + b\Delta x + c = f(x_{n+1})$$

$$g(0) = c = f(x_n)$$

$$2a\Delta x^2 + 2c = f(x_{n-1}) + f(x_{n+1})$$

$$\text{Area} = \int_{-\Delta x}^{\Delta x} (ax^2 + bx + c)\,dx = \frac{ax^3}{3} + \frac{bx^2}{2} + cx \Big|_{-\Delta x}^{\Delta x}$$

$$= a\frac{\Delta x^3}{3} + b\frac{\cancel{\Delta x^2}}{2} + c\Delta x + a\frac{\Delta x^3}{3} - b\frac{\cancel{\Delta x^2}}{2} + c\Delta x$$

$$= \frac{2a\Delta x^3}{3} + 2c\Delta x = \frac{2a\Delta x^3}{3} + \frac{6c\Delta x}{3}$$

$$= \frac{2a\Delta x^2 + 2c + 4c}{3}\Delta x$$

$$= \frac{f(x_{n-1}) + f(x_{n+1}) + 4f(x_n)}{3}\Delta x$$

Total area $= \Big[[f(x_0) + 4f(x_1) + f(x_2)] + [f(x_2) + 4f(x_3) + f(x_4)]$
$+ \dots + [f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})]$
$+ [f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \Big] \cdot \frac{\Delta x}{3}$

$= [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots$
$+ 2f(x_{n-4}) + 4f(x_{n-3}) + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \frac{\Delta x}{3}$

Example: $\int_1^2 \frac{1}{x}\,dx$ $\quad n=8$

i	x_i	$f(x_i)$	wef	prod
0	1	1	1	1
1	$\frac{9}{8}$	$\frac{8}{9}$	4	$\frac{32}{9}$
2	$\frac{5}{4}$	$\frac{4}{5}$	2	$\frac{8}{5}$
3	$\frac{11}{8}$	$\frac{8}{11}$	4	$\frac{32}{11}$
4	$\frac{3}{2}$	$\frac{2}{3}$	2	$\frac{4}{3}$
5	$\frac{13}{8}$	$\frac{8}{13}$	4	$\frac{32}{13}$
6	$\frac{7}{4}$	$\frac{4}{7}$	2	$\frac{8}{7}$
7	$\frac{15}{8}$	$\frac{8}{15}$	4	$\frac{32}{15}$
8	2	$\frac{1}{2}$	1	$\frac{1}{2}$

Sum = 16.6357087357

$\times \frac{1}{3} \times \frac{1}{8} = .6931545306$6

$\int_1^2 \frac{1}{x}\,dx = \ln 2 \approx .6931471805$6

Diff = .0000073500$95

Theorem

Trap. Rule: If f'' is continuous and $|f''(x)| \le M$ on $[a,b]$

Then the error in trap. rule satisfies

$$|E_T| \le \frac{M(b-a)^3}{12n^2}$$

Simp. Rule: If $f^{(4)}$ is continuous and $|f^{(4)}(x)| \le M$ on $[a,b]$ then the error in Simpson's Rule satisfies $|E_S| \le \frac{M(b-a)^5}{180n^4}$

Example: $f(x) = \ln x$ $\quad f'''(x) = \frac{2}{x^3}$

$f'(x) = \frac{1}{x}$ $\quad f^{(4)}(x) = \frac{-6}{x^4}$

$f''(x) = -\frac{1}{x^2}$

$$|f^{(4)}(x)| \le 6 \quad \text{on } [1,2]$$

For $\int_1^2 \frac{1}{x}\,dx$, $\quad |E_S| \le \frac{6(1)}{180n^4} = \frac{1}{30n^4}$

If we want $|E_S| \le 10^{-8}$, need

$$\frac{1}{30n^4} \le 10^{-8}$$

so $\frac{10^8}{30} \le n^4$

$$42.7 \approx \frac{10^2}{\sqrt[4]{30}} \le n$$

so need $n = \underline{\underline{44}}$