8.7 Improper Integrals

$$\int_a^\infty f(x)\,dx = \lim_{A\to\infty}\int_a^A f(x)\,dx$$

Example: $$\int_0^\infty e^{-x}\,dx = \lim_{A\to\infty}\int_0^A e^{-x}\,dx = \lim_{A\to\infty} -e^{-x}\Big|_0^A$$

$$= \lim_{A\to\infty}\left[-e^{-A} - (-e^0)\right] = \lim_{A\to\infty} 1 - e^{-A} = 1$$

Example: $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{A\to\infty}\int_1^A \frac{1}{x^2}\,dx = \lim_{A\to\infty} -\frac{1}{x}\Big|_1^A$$

$$= \lim_{A\to\infty} -\frac{1}{A} - \left(-\frac{1}{1}\right) = \lim_{A\to\infty} 1 - \frac{1}{A} = 1$$

Example: $$\int_1^\infty \frac{1}{x}\,dx = \lim_{A\to\infty}\int_1^A \frac{1}{x}\,dx = \lim_{A\to\infty} \ln|x|\Big|_1^A$$

$$= \lim_{A\to\infty} (\ln A - \ln 1) = \lim_{A\to\infty} \ln A = \infty$$ (DIVERGES)

Example: $$\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{A\to-\infty}\int_A^0 \frac{1}{1+x^2}\,dx = \lim_{A\to-\infty} \tan^{-1}x\Big|_A^0$$

$$= \lim_{A\to-\infty} \tan^{-1}0 - \tan^{-1}A = \pi/2$$

Example: $$\int_{-\infty}^\infty e^x\,dx = \lim_{A\to-\infty}\int_A^0 e^x\,dx + \lim_{A\to\infty}\int_0^A e^x\,dx$$

$$= \lim_{A\to-\infty} e^x\Big|_A^0 + \lim_{A\to\infty} e^x\Big|_0^A$$

$$= \lim_{A\to-\infty} 1 - e^A + \lim_{A\to\infty} e^A - 1 = \infty$$

Discontinuities:

Example

$$\int_0^1 \frac{1}{x}\,dx = \lim_{A\to 0^+} \int_A^1 \frac{1}{x}\,dx$$

$$= \lim_{A\to 0^+} \ln|x| \Big|_A^1 = \lim_{A\to 0^+} -\ln A = \infty$$

DIVERGES

Example

$$\int_{-1}^{1} x^{-1/3}\,dx = \lim_{A\to 0^-} \int_{-1}^{A} x^{-2/3}\,dx + \lim_{A\to 0^+} \int_A^1 x^{-2/3}\,dx$$

$$= \lim_{x\to 0^-} 3x^{2/3}\Big|_{-1}^{A} + \lim_{A\to 0^+} 3x^{2/3}\Big|_A^1$$

$$= \lim_{x\to 0^-} 3A^{2/3} - 3 + \lim_{A\to 0^+} 3 - 3A^{2/3}$$

$$= -3 + 3 = 0$$

CONVERGES

Tests for Convergence

Comparison Test: If f & g are continuous on $[a,\infty)$ and $0 \le f(x) \le g(x)$ on $[a,\infty)$ then

1) If $\int_a^\infty g(x)\,dx$ converges then $\int_a^\infty f(x)\,dx$ also converges

(2) If $\int_a^\infty f(x)\,dx$ diverges then $\int_a^\infty g(x)\,dx$ also diverges.

Example: $\int_1^\infty \sin x\, e^{-x}\,dx$ converges because on $[1,\infty)$, $\sin x\, e^{-x} \le e^{-x}$ and $\int_1^\infty e^{-x}\,dx$ converges.

Example: $\int_1^\infty \frac{x^2+1}{x}\,dx$ diverges because on $[1,\infty)$, $\frac{x^2+1}{x} \ge \frac{1}{x}$ and $\int_1^\infty \frac{1}{x}\,dx$ diverges.

Limit Comparison Test:

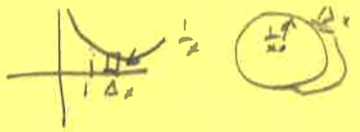
If f & g are continuous on $[a,\infty)$ and $f(x) \geq 0$ and $g(x) \geq 0$ on $[a,\infty)$ and if $\lim_{x\to\infty} \frac{f(x)}{g(x)} > 0$ (and the limit exists - not ∞) then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge.

Example: $\int_1^\infty \frac{1}{x^2-x+1}\,dx$ compare to $\int_1^\infty \frac{1}{x^2}\,dx$

Cannot use comparison test because sometimes $\frac{1}{x^2-x+1} \geq \frac{1}{x^2}$ which converges. Limit comparison: $\lim_{x\to\infty} \frac{\frac{1}{x^2-x+1}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{x^2-x+1}$

$= \lim_{x\to\infty} \frac{1}{1-\frac{1}{x}+\frac{1}{x^2}} = 1.$ Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges, so does $\int_1^\infty \frac{1}{x^2-x+1}\,dx$.

Examples: the graph of $y = \frac{1}{x}$, x in $[1,\infty)$ is rotated around the x-axis. Find the volume and surface area of the solid.



Volume of disk $= \pi\left(\frac{1}{x^2}\right)\Delta x$

total volume $= \int_1^\infty \frac{\pi}{x^2}\,dx = \lim_{A\to\infty}\int_1^A \frac{\pi}{x^2}\,dx$

$$= \lim_{A\to\infty} -\frac{\pi}{x}\Big|_1^A = \lim_{A\to\infty} -\frac{\pi}{A} - (-1) = 1$$

$y'^2 = \left(-\frac{1}{x^2}\right)^2 = \frac{1}{x^4}$ $\quad SA = 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$

$\frac{1}{x}\sqrt{1+\frac{1}{x^4}} \geq \frac{1}{x}$ on $[1,\infty)$ and $\int_1^\infty \frac{1}{x}\,dx$ diverges, so

$2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$ also diverges.