

Calculus II Comprehensive Review Sheet

Integration Techniques, Physical Applications & Differential Equations

1 Integration by Parts

Core Formula: Derived from the product rule for differentiation.

$$\int u \, dv = uv - \int v \, du \quad \text{or for definite integrals:} \quad \int_a^b u \, dv = [uv]_a^b - \int_a^b v \, du$$

Strategy for Selecting u and dv (LIATE Rule): Choose u in order of priority from top to bottom. The remaining factor becomes dv .

1. **L** – Logarithmic functions ($\ln x, \log_b x$)
2. **I** – Inverse Trigonometric functions ($\arcsin x, \arctan x$)
3. **A** – Algebraic functions ($x^n, \sqrt{x}, 3x^2 + 1$)
4. **T** – Trigonometric functions ($\sin x, \cos x, \sec^2 x$)
5. **E** – Exponential functions ($e^x, 2^x$)

Special Techniques:

- **Repeated Parts:** Used when integrating polynomials multiplied by exponential or trigonometric functions (e.g., $\int x^2 e^{3x} dx$). Reduce the degree of the polynomial step-by-step.
- **Cyclic / Recurring Integrals:** When integrating products of exponentials and trig functions (e.g., $\int e^x \cos x dx$), apply IBP twice to regenerate the original integral I , then solve algebraically for I .

2 Work Applications

General Definition of Work: $W = \int_a^b F(x) \, dx$ where $F(x)$ is the force exerted along a path.

2.1 1. Hooke's Law & Springs

- **Hooke's Law Force:** $F(x) = kx$, where k is the spring constant (N/m or lb/ft) and x is the **displacement from natural length**.
- **Work to Stretch/Compress from x_1 to x_2 :**

$$W = \int_{x_1}^{x_2} kx \, dx = \frac{1}{2}k(x_2^2 - x_1^2)$$

- **Caution:** Always measure x relative to the natural length ($x = \text{Current Length} - \text{Natural Length}$).

2.2 2. Pumping Fluid from a Tank

To calculate work required to empty a tank by pumping fluid to a height h_{exit} :

$$W = \int_{y_{\min}}^{y_{\max}} \text{Force}(y) \times \text{Distance}(y) = \int_{y_{\min}}^{y_{\max}} [\rho \cdot g \cdot A(y) \cdot dy] \cdot d(y)$$

- $\rho \cdot g$ = Weight Density (δ). For water: $9800 \, N/m^3$ (Metric) or $62.4 \, lb/ft^3$ (US Customary).
- $A(y) \, dy$ = Volume of a thin horizontal slice at height y .
- $d(y)$ = Distance the slice at height y must be lifted to reach the exit level.

Common Tank Cross-Section Areas $A(y)$:

Tank Shape	Width / Radius at Height y	Area $A(y)$
Rectangular Prism	Constant width W , length L	$A(y) = L \cdot W$
Cylinder (Vertical)	Constant radius R	$A(y) = \pi R^2$
Cone (Apex Down, Radius R , Height H)	Similar Triangles: $r(y) = \frac{R}{H}y$	$A(y) = \pi \left(\frac{R}{H}y\right)^2$
Sphere (Radius R , centered at origin)	$r(y) = \sqrt{R^2 - y^2}$	$A(y) = \pi(R^2 - y^2)$

3 Centroids of Planar Regions

The centroid $(\bar{x}, \bar{y})$ represents the geometric center of mass of a thin plate with uniform density.

3.1 Region Bounded Between $y = f(x)$ and $y = g(x)$ with $f(x) \geq g(x)$ on $[a, b]$

- **Total Area (A):**

$$A = \int_a^b [f(x) - g(x)] dx$$

- **Moment about y -axis (M_y):**

$$M_y = \int_a^b x [f(x) - g(x)] dx$$

- **Moment about x -axis (M_x):**

$$M_x = \frac{1}{2} \int_a^b [(f(x))^2 - (g(x))^2] dx$$

- **Centroid Coordinates:**

$$\bar{x} = \frac{M_y}{A}, \quad \bar{y} = \frac{M_x}{A}$$

3.2 Symmetry Principle

If a region is symmetric about a line L , its centroid must lie on L . For example, if a region is symmetric about the y -axis, then $\bar{x} = 0$.

4 First-Order Differential Equations

4.1 1. Separable Differential Equations

An equation of the form $\frac{dy}{dx} = g(x)h(y)$.

1. **Separate variables:** Multiply/divide to collect y terms on the left and x terms on the right:

$$\frac{1}{h(y)} dy = g(x) dx$$

2. **Integrate both sides:** $\int \frac{1}{h(y)} dy = \int g(x) dx + C$.
3. **Solve for y explicitly** if possible, and apply initial conditions $y(x_0) = y_0$ to find C .

4.2 2. First-Order Linear Differential Equations

Standard Form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Solution Procedure (Integrating Factor Method):

1. Ensure the coefficient of $\frac{dy}{dx}$ is 1.
2. Compute the **Integrating Factor**:

$$\mu(x) = e^{\int P(x) dx} \quad (\text{Note: drop constant } C \text{ here})$$

3. Multiply the entire standard equation by $\mu(x)$:

$$\frac{d}{dx} [\mu(x)y] = \mu(x)Q(x)$$

4. Integrate both sides with respect to x :

$$\mu(x)y = \int \mu(x)Q(x) dx + C \implies y(x) = \frac{1}{\mu(x)} \left[\int \mu(x)Q(x) dx + C \right]$$

5 Applied Differential Equations

5.1 1. Newton's Law of Heating and Cooling

The rate of change of temperature $T(t)$ is proportional to the difference between $T(t)$ and ambient temperature T_s :

$$\frac{dT}{dt} = k(T - T_s) \quad \text{or} \quad \frac{dT}{dt} = -k(T - T_s) \quad (k > 0)$$

5.2 2. Tank Mixing Problems

Let $y(t)$ be the amount of salt (or solute) in the tank at time t .

$$\frac{dy}{dt} = \text{Rate In} - \text{Rate Out}$$

- **Rate In:** (Concentration In) $\times$ (Flow Rate In) $= C_{\text{in}} \cdot r_{\text{in}}$
- **Rate Out:** (Concentration Out) $\times$ (Flow Rate Out) $= \left(\frac{y(t)}{V(t)} \right) \cdot r_{\text{out}}$
- **Volume of Liquid $V(t)$:**

$$V(t) = V_0 + (r_{\text{in}} - r_{\text{out}})t$$

- If $r_{\text{in}} = r_{\text{out}}$, volume is constant: $V(t) = V_0$.
- (Not on exam!) If $r_{\text{in}} \neq r_{\text{out}}$, volume varies with time.

$$\frac{dy}{dt} + \left(\frac{r_{\text{out}}}{V_0 + (r_{\text{in}} - r_{\text{out}})t} \right) y = C_{\text{in}} \cdot r_{\text{in}}$$