Complex Numbers

Complex numbers are very important for the solution of differential equations. Even when the solutions are real, complex numbers are used in the solution methods.

What is a Complex Number?

Definition. A *complex number* is a number of the form $z = a + bi$, where a and b are real numbers and $i^2 = -1$. The *real part* of $a + bi$ is a, while the *imaginary part* is b.

$z = 2 + 3i$	$a = 2$	$b = 3$
$z = 1 - i$	$a = 1$	$b = -1$
$z = 4i$	$a = 0$	$b = 4$
$z = 16$	$a = 16$	$b = 0$

Notation. We write $\mathrm{Re}(z)$ for the real part of a complex number z and $\mathrm{Im}(z)$ for the imaginary part.

$z = 2 + 3i$	$\mathrm{Re}(z) = 2$	$\mathrm{Im}(z) = 3$
$z = 1 - i$	$\mathrm{Re}(z) = 1$	$\mathrm{Im}(z) = -1$
$z = 3$	$\mathrm{Re}(z) = 3$	$\mathrm{Im}(z) = 0$
$z = 2i$	$\mathrm{Re}(z) = 0$	$\mathrm{Im}(z) = 2$

Imaginary Numbers

Definition. An *imaginary number* is a complex number whose real part is 0.

Observation. A real number is a complex number whose imaginary part is 0.

Why Do We Need Complex Numbers?

The real numbers suffer from an unfortunate defect – not every polynomial equation has a real solution. For example, $x^2 + 1 = 0$ has no real solutions, because for any real number x, $x^2 \geq 0$, so $x^2 + 1 \geq 1 > 0$. By extending the real numbers with i to get the complex numbers, we immediately get a solution to the equation $x^2 + 1 = 0$, since $i^2 + 1 = -1 + 1 = 0$.

Complex Arithmetic

The remarkable fact (which is not at all obvious) is that we also get solutions to *every* other polynomial equation as well.

We can do the usual arithmetic operations, addition, subtraction, multiplication, and division, complex numbers as well as real numbers. Addition, subtraction, and multiplication are fairly straightforward. Division is somewhat more complicated, and requires the idea of a complex conjugate. We'll also define the magnitude of a complex number, which extends the idea of absolute value.

Complex Addition and Subtraction

To add or subtract complex numbers, we just add or subtract the real parts and imaginary parts separately.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a + c) - (b + d)i$$

Example. $(2+3i)+(4-5i)=(2+4)+(3-5)i=6-2i$

Example. $(2+3i)-(4-5i)=(2-4)+(3+5i)=-2+8i$

Example. $(2+3i)-i=(2+0)+(3-1)i=2+2i$

Example. $(2+3i)+4=(2+4)+(3+0)i=6+3i$

Complex Multiplication

To multiply two complex numbers, we pretend that they are polynomials and that i is a variable, multiply them out (FOIL), replace i^2 by -1, and simplify.

$$\begin{aligned}(a+bi)(c+di) &= ac+adi+bci+bdi^2\\ &= ac+adi+bci+bd(-1)\\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

Learn the technique – not the formula!

Example. $(2+3i)(4-5i)=8-10i+12i-15i^2=8-10i+12i-15(-1)=23+2i$

Example. $(2+3i)(5)=10+15i$

Example. $(2+3i)(-i)=-2i-3i^2=-2i-3(-1)=3-2i$

The Complex Conjugate

Definition. If $z=a+bi$ is a complex number with a and b real, then the *complex conjugate* of z is $a-bi$. We just negate the imaginary part.

Notation. The notation for the complex conjugate of z is $\overline{z}$.

Observations.

1. If z is real, then $\overline{z}=z$.
2. If z is imaginary, then $\overline{z}=-z$.

Example. $\overline{2+3i}=2-3i$

Example. $\overline{-3-2i}=-3+2i$

Example. $\overline{4}=4$

Example. $\overline{i}=-i$

Complex Division

In order to divide and reduce the answer to the standard form $c + di$, we'll employ a variation on a familiar trick – multiply top and bottom by the complex conjugate of the bottom.

$$\begin{aligned} \frac{a+bi}{c+di} &= \frac{(a+bi)(c-di)}{(c+di)(c-di)} \\ &= \frac{ac - adi + bci - bdi^2}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i \end{aligned}$$

Learn the technique – not the formula!

Example. $\dfrac{2+3i}{3-4i} = \dfrac{(2+3i)(3+4i)}{(3-4i)(3+4i)}$

Example. $\dfrac{6+8i+9i-12}{9+16} = -\dfrac{6}{25} + \dfrac{17}{25} i$

Example. $\dfrac{2+3i}{5} = \dfrac{2}{5} + \dfrac{3}{5} i$

Example. $\dfrac{1}{i} = \dfrac{(1)(-i)}{(i)(-i)} = \dfrac{-i}{1} = -i$

Magnitude

Definition. The *magnitude* of the complex number z is $\sqrt{z\bar{z}}$.

Notation. The notation for the magnitude of the complex number c is $|c|$ (like absolute value).

We'll derive a simple formula for the magnitude.

$$\begin{aligned} |a+bi| &= \sqrt{(a+bi)(a-bi)} \\ &= \sqrt{a^2 + abi - abi - b^2 i^2} \\ &= \sqrt{a^2 - b^2(-1)} \\ &= \sqrt{a^2+b^2} \end{aligned}$$

Memorize this formula: $|a+bi| = \sqrt{a^2+b^2}$

Example. $|2+3i| = \sqrt{2^2+3^2} = \sqrt{13}$

Example. $|5-12i| = \sqrt{5^2+12^2} = 13$

Example. $|-12| = 12$

Example. $|-3i| = 3$

Observations.

1. Since $a^2 + b^2 \geq 0$, we can always take the square root, so magnitude is defined for all complex numbers.
2. For real numbers (which are also complex numbers) it appears that there might be confusion about the meaning of $|\ |$. However, if r is real,

$$|r| = \sqrt{r\overline{r}} = \sqrt{r^2} = |r|$$

   The first absolute means complex magnitude, while the second means ordinary (real) absolute value. Thus the magnitude of a real number is its absolute value, so there is no problem in using the same notation.

3. If $z = a + bi$ with a and b real, then $|z|^2 = a^2 + b^2$, which is always a nonnegative real number.

Complex Roots of Polynomials

Definition. The *degree* of a polynomial is the highest power of a variable that actually appears (with a nonzero coefficient).

Example. $x^4 - 5x^2 - 1$ has degree 4.

Example. $2x + 3$ has degree 1.

Example. 4 has degree 0 (for constants, there is an implicit x^0).

Definition. A *root* of a polynomial $p(x)$ is a solution of the equation $p(x) = 0$; i.e., a number that can be substituted into the polynomial to get 0.

Example. $x^2 - 4$ has roots 2 and -2,

Example. $x + 5$ has only one root, -5.

Example. $x^2 + 1$ has roots i and $-i$.

The following theorem gives the all-important relationship between roots and factors.

Theorem. The number r is a root of a polynomial $p(x)$ if and only if $x - r$ is a factor of $p(x)$.

Example. $x^2 - 3x + 2 = (x - 1)(x - 2)$ and has roots 1 and 2.

Example. $x^2 + 1 = (x + i)(x - i)$ and has roots i and $-i$.

Multiplicities

Definition. If a polynomial has $(x - r)^k$ as a factor and this is the largest power of $x - r$ that is a factor, then we say that r is a *root of multiplicity* k.

Example. $x^3 - 2x^2 + x = x(x - 1)^2$. The roots are 0 (multiplicity 1) and 1 (multiplicity 2).

Example. $x^4 - 8x^2 + 16 = (x - 2)^2(x + 2)^2$. The roots are 2 and -2 (both multiplicity 2).

The Fundamental Theorem of Algebra

Theorem. Every polynomial with real or complex coefficients factors completely using complex numbers, and so the sum of the multiplicities of the complex roots of a polynomial of degree n is n.

Many polynomials with real coefficients have no real roots, but they do have complex roots.

Example. $x^4 - 1 = (x-i)(x+i)(x-1)(x+1)$. The roots are 1, -1, i, and $-i$, each of multiplicity 1. The sum of the multiplicities is 4.

Example. $x^4 + 2x^2 + 1 = (x+i)^2(x-i)^2$. The roots are i and $-i$, each of multiplicity 2. The sum of the multiplicities is 4.

The Quadratic Formula With Complex Numbers

To find complex roots of the quadratic equation $ax^2 + bx + c = 0$ using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

we just plug in the coefficients. If the discriminant $(b^2 - 4ac)$ is negative, there are no real roots, but there are two complex roots. We replace the radical by an equivalent imaginary number.

Example. Find the roots of $x^2 + x + 1$.

Solution.

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} \\ &= \frac{-1}{2} \pm \frac{\sqrt{-3}}{2} \\ &= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \end{aligned}$$

Completing the Square With Complex Numbers

We can also solve quadratic equations by completing the square, and this is often easier than using the quadratic formula. Recall that to complete a square, take half of the middle coefficient, square it, and split this quantity off from the constant term to give a perfect square.

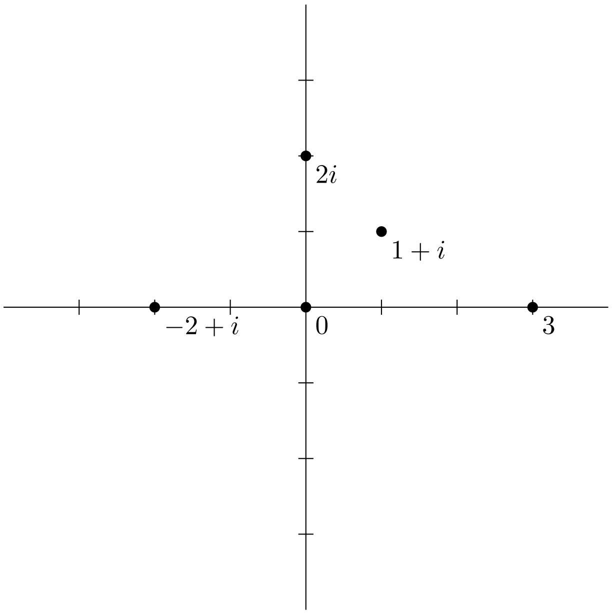
Example. Find the roots of $x^2 + x + 1$.

Solution.

$$\begin{aligned} x^2 + x + 1 &= 0 \\ x^2 + x + \frac{1}{4} + \frac{3}{4} &= 0 \\ \left(x + \frac{1}{2}\right)^2 &= -\frac{3}{4} \\ x + \frac{1}{2} &= \sqrt{-\frac{3}{4}} \\ x &= -\frac{1}{2} \pm \frac{\sqrt{-3}}{2} \\ &= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \end{aligned}$$

The Complex Plane

We can display complex numbers graphically using the complex plane. We plot numbers using the imaginary part for the y coordinate and the real part for the x coordinate. The magnitude of the complex number is its distance from the origin.



Euler's Formula

Definition. The complex exponential e^z where z is a complex number is defined by the power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

This series converges for all real and complex values of x. The sin and cos functions for complex numbers are defined in a similar fashion, using the same power series coefficients as in the series for real numbers.

By substituting ix for z in the power series, we get the following formula:

Euler's Formula

$$e^{ix} = \cos x + i \sin x$$

Example. $e^0 = \cos 0 + i \sin 0 = 1$

Example. $e^{i\pi} = \cos \pi + i \sin \pi = -1$

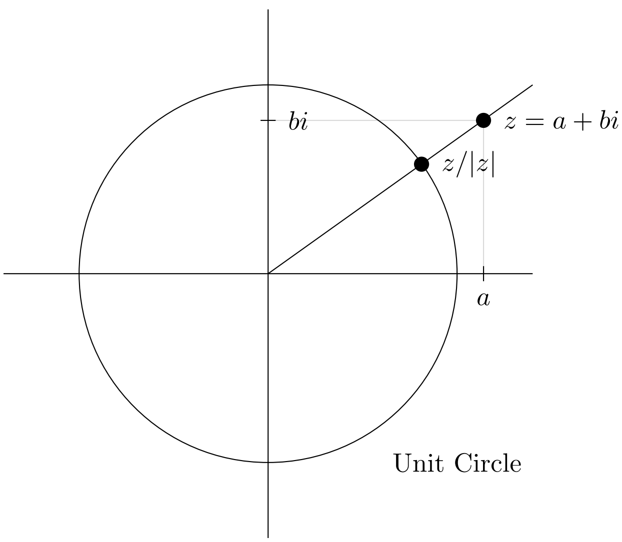
Example. $e^{i\pi/4} = \cos \pi/4 + i \sin \pi/4 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$

Example. $e^{2+7i\pi/6} = e^2 e^{7i\pi/6} = e^2(\cos 7\pi/6 + i \sin 7\pi/6) = e^2 \left(-\frac{\sqrt{3}}{2} + \frac{1}{2} i\right)$

Complex Numbers in Polar Form

Here are some important facts:

1. If $z = a + bi$ is a complex number, then $\dfrac{z}{|z|}$ has magnitude 1. Thus in the complex plane, it lies on the unit circle.
2. If θ is the angle from the positive x-axis to $\dfrac{z}{|z|}$, then $\cos\theta = \operatorname{Re}\left(\dfrac{z}{|z|}\right) = \dfrac{a}{a^2 + b^2}$ and $\sin\theta = \operatorname{Im}\left(\dfrac{z}{|z|}\right) = \dfrac{b}{a^2 + b^2}$.
3. Thus $\dfrac{z}{|z|} = e^{i\theta}$, and so $z = |z|e^{i\theta}$.
4. This is called the *polar form* of z.
5. $\tan\theta = \dfrac{b}{a}$ (unless $a = 0$).
6. If $a > 0$, then $\theta = \tan^{-1} b/a$.
7. If $a < 0$, then $\theta = \pi + \tan^{-1} b/a$.
8. If $a = 0$ and $b > 0$, then $\theta = \pi/2$.
9. If $a = 0$ and $b < 0$ then $\theta = 3\pi/2$.



We can use these facts to convert from rectangular form to polar form.

Example. Express $z = -2 + 2\sqrt{3}i$ in polar form.

Solution. First we find the magnitude: $|z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = 4$. Next we find the angle. $\tan\theta = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$. It appears that $\theta = \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$, but this isn't quite right because $-\frac{\pi}{3}$ is in the fourth quadrant. Since the real part is negative and the imaginary part is positive, we need an angle in the second quadrant. Therefore we add π to get $\frac{2\pi}{3}$. Thus the polar form is $4\left(\cos\frac{2\pi}{3} + \sin\frac{2\pi}{3}i\right) = 4e^{2\pi i/3}$.