Partial Derivatives

If f is a function of more than one variable, then a *partial derivative* of f is a derivative taken with respect to one variable while holding all of the other variables constant. The notation for the partial derivative of f with respect to x is $\frac{\partial f}{\partial x}$ or f_x. For example, if $f(x,y) = x^2 \sin y$, then:

$$\frac{\partial f}{\partial x} = 2x \sin y \qquad\qquad \frac{\partial f}{\partial y} = x^2 \cos y$$

We just pretend that y is a constant and differentiate with respect to x, or vice versa.

If we want to emphasize the arguments to the partial derivative function, we can write $\frac{\partial f}{\partial x}(x,y)$ or $f_x(x,y)$. This notation is particularly important if we want to evaluate the partial for particular values of x and y. For the function above, $\frac{\partial f}{\partial x}\left(\frac{1}{2}, \frac{\pi}{2}\right) = 2\left(\frac{1}{2}\right) \sin \frac{\pi}{2} = 1$.

All of the formulas for computing ordinary derivatives also work for partial derivatives; e.g., the product and quotient formulas and the chain rule.

The actual definition of partial derivatives is in terms of limits:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad\qquad \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

However, we rarely need to resort to the definition to compute partial derivatives.

Higher Order Partials

The second partial of f with respect to x is just the partial of $\frac{\partial f}{\partial x}$ with respect to x. The notation is:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad\qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

We can also write f_{xx} for the second partial of f with respect to x. We can also calculate third and higher partials, but there is rarely a need for partials of order higher than 2.

Mixed Partials

Mixed partials are taken first with respect to one variable and then with respect to another.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \qquad\qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

The alternate notation is $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$ and $\frac{\partial^2 f}{\partial y \partial x} = f_{xy}$. Note the order of the x's and y's. The notation makes sense when you think of it as $f_{xy} = (f_x)_y$ and $f_{yx} = (f_y)_x$. An important theorem of multivariable calculus says that if the two mixed partials are continuous, then they are equal, so most of the time, the order doesn't matter.

For the function in the example above,

$$\frac{\partial^2 f}{\partial x^2} = 2 \sin y \qquad \frac{\partial^2 f}{\partial y^2} = -x^2 \sin y \qquad \frac{\partial^2 f}{\partial x \partial y} = 2x \cos y \qquad \frac{\partial^2 f}{\partial y \partial x} = 2x \cos y$$